

Transportation Cost Inequality on Path Spaces with Uniform Distance ^{*}

Shizan Fang^{a,b}, Feng-Yu Wang^{a,c†}, Bo Wu^a

^a School of Mathematical Science & Lab. Math. Com. Sys.,

Beijing Normal University, Beijing 100875, China

^b I.M.B. B.P. 47870, Université de Bourgogne, Dijon, France

^c WIMCS, University of Wales Swansea, Singleton Park, Swansea, SA2 8 PP, UK

February 2, 2008

Abstract

Let M be a complete Riemannian manifold and μ the distribution of the diffusion process generated by $\frac{1}{2}\Delta + Z$ where Z is a C^1 -vector field. When $\text{Ric} - \nabla Z$ is bounded below and Z has, for instance, linear growth, the transportation-cost inequality with respect to the uniform distance is established for μ on the path space over M . A simple example is given to show the optimality of the condition.

AMS subject Classification: 60J60, 58J60

Keyword: Transportation cost inequality, path space, damped gradient, quasi-invariant flow, uniform distance.

1 Introduction

Since Talagrand [16] found his transportation cost inequality for the Gaussian measure on \mathbb{R}^d , this inequality have been established on finite- and infinite-dimensional spaces with respect to many different distances (i.e. cost-functions); see [18] for historical comments and

^{*}Supported in part by NNSFC(10121101) and RFDF(20040027009)

[†]wangfy@bnu.edu.cn

references. For instance, on the path space of a diffusion process on a complete Riemannian manifold, the inequality holds with respect to the L^2 -distance provided the curvature of the diffusion is bounded below, and it holds with respect to the intrinsic distance induced by the Malliavian gradient provided the curvature is bounded (see [19, 20]). See also [10, 22, 23] for the study of diffusion path spaces over \mathbb{R}^d , and [8, 15] for the study on path and loop groups. The purpose of this paper is to search for a reasonable curvature condition such that the Talagrand inequality holds for the distribution of the corresponding diffusion process with respect to the uniform distance on the path space.

Let (M, g) be a connected complete Riemannian manifold of dimension d . Consider the diffusion operator $L = \frac{1}{2}\Delta_M + Z$ for a C^1 -vector field Z . Assume that

$$(1.1) \quad \text{Ric} - \nabla Z \geq -K.$$

Let $o \in M$ and $T > 0$ be fixed. Let $H : TM \rightarrow TO(M)$ the horizontal lift, where $O(M)$ is the orthonormal frame bundle over M . Consider the stochastic differential equation on $O(M)$:

$$(1.2) \quad du_t(w) = \sum_{i=1}^d H_i(u_t(w)) \circ dw_t^i + \frac{1}{2} H_Z(u_t(w)) dt, \quad u_0 \in \pi^{-1}(o),$$

where $w_t = (w_t^i : 1 \leq i \leq d)$ is the Brownian motion on \mathbb{R}^d and $H_i(u) := H_{ue_i}$, $1 \leq i \leq d$. Here and in what follows, $\{e_i\}_{i=1}^d$ is the canonical orthonormal basis on \mathbb{R}^d . Let

$$W_0(\mathbb{R}^d) := \{w \in C([0, T]; \mathbb{R}^d) : w_0 = 0\},$$

$$\mathbb{H} := \left\{ h \in W_0(\mathbb{R}^d) : \|h\|_{\mathbb{H}}^2 := \int_0^T \dot{h}_s^2 ds < \infty \right\},$$

and \mathbb{P} is the standard Wiener measure. Let $\pi : O(M) \rightarrow M$ be the canonical projection. Then $\gamma_t(w) := \pi u_t(w)$ is the L -diffusion process on M starting from o , which is non-explosive under the condition (1.1). Let μ be the law of $w \mapsto \gamma(w) \in W_o(M) := \{\gamma \in C([0, T]; M) : \gamma_0 = o\}$.

Let ρ be the Riemannian distance on M . For two probability measures μ_1, μ_2 on $W_o(M)$, let $W_{2,d_\infty}^2(\mu_1, \mu_2)$ be the L^2 -Wasserstein distance between them induced by the uniform norm

$$d_\infty(\gamma, \eta) := \sup_{t \in [0, T]} \rho(\gamma_t, \eta_t), \quad \gamma, \eta \in W_o(M).$$

More precisely,

$$(1.3) \quad W_{2,d_\infty}^2(\mu_1, \mu_2) := \inf_{\hat{\pi} \in \mathcal{C}(\mu_1, \mu_2)} \int_{W_o(M) \times W_o(M)} d_\infty^2(\gamma, \eta) \hat{\pi}(d\gamma, d\eta)$$

where $\mathcal{C}(\mu_1, \mu_2)$ is the set of all probability measures on $W_o(M) \times W_o(M)$ with marginal distributions μ_1 and μ_2 . The main result of the paper is the following:

Theorem 1.1. Assume (1.1) hold for some $K \geq 0$ and let $\rho_o = \rho(o, \cdot)$. If $|Z| \leq \psi \circ \rho_o$ for some strictly positive and increasing $\psi \in C^\infty([1, \infty))$ with

$$\int_0^\infty \frac{1}{\psi(s)} ds = \infty,$$

then

$$(1.4) \quad W_{2, d_\infty}^2(F\mu, \mu) \leq 2 \frac{e^{KT} - 1}{K} \mu(F \log F), \quad F \geq 0, \mu(F) = 1.$$

To prove this result, we could start from the log-Sobolev inequality for damped gradients \tilde{D} (2.7) below. To this end, one would like to follow the line of [3] by studying the Hamilton-Jacobi semigroup Q_t induced by the uniform norm d_∞ :

$$(Q_t F)(\gamma) = \inf_{\eta \in W_o(M)} \left\{ F(\eta) + \frac{1}{2t} d_\infty^2(\gamma, \eta) \right\}.$$

By [15], Q_t preserves the class of d_∞ -Lipschitz functions. Then, according to the argument of [3], to derive the desired transportation cost inequality from the log-Sobolev inequality (2.7), it remains to prove that

$$D_t^+ Q_t F := \limsup_{s \downarrow 0} \frac{Q_{t+s} F - Q_t F}{s} \leq -C \int_0^T |\tilde{D}_s Q_t F|^2 ds$$

for some constant $C > 0$, the inequality for which we are actually in position to prove if $Z = 0$.

So, in this paper we shall follow the line of [20] using finite-dimensional approximations. To make the corresponding finite approximate metric continuous, we have to first assume that the Ricci curvature is C_b^1 , the curvature tensor Ω is C_b^0 and the drift is C_b^2 . So, to finish the prove, we adopt one more approximation argument on the Riemannian metric and the drift to fit the above regularity assumption. To realize the second approximation procedure we need the growth condition of $|Z|$ stated in Theorem 1.1. On the other hand, however, since the growth of $|Z|$ is not included in the inequality (1.4), we believe that it is technical rather than necessary.

To conclude this section, we present below a simple example to show that the condition in Theorem 1.1 for (1.4) is sharp.

Example 1.1. Let $M = \mathbb{R}^d$ and $Z = \nabla V$ for

$$V(x) := (1 + |x|^2)^\delta, \quad x \in \mathbb{R}^d,$$

where $\delta \geq 0$ is a constant. Let $T > 0$ and $o = 0 \in \mathbb{R}^d$ be fixed. We claim that there exists a constant $C > 0$ such that

$$(1.5) \quad W_{2,d_\infty}^2(F\mu, \mu) \leq C\mu(F \log F), \quad F \geq 0, \mu(F) = 1$$

holds if and only if either $\delta \leq 1$. Indeed, for $\delta \leq 1$ $\text{Ric} - \nabla Z = -\text{Hess}_V$ is bounded from below and $|Z|$ has at most linear growth. So, (1.5) follows from Theorem 1.1. On the other hand, it is well-known that (1.5) implies

$$(1.6) \quad \mathbb{E} \exp \left[\lambda \sup_{t \in [0, T]} |\gamma_t|^2 \right] < \infty$$

holds for some $\lambda > 0$, where γ_t is the L -diffusion process starting from 0. Indeed, according to [22], this concentration property is equivalent to the weaker L^1 transportation cost inequality:

$$W_{1,d_\infty}^2(F\mu, \mu) \leq C\mu(F \log F), \quad F \geq 0, \mu(F) = 1$$

for some constant $C > 0$, where

$$W_{1,d_\infty}(\mu_1, \mu_2) := \inf_{\hat{\pi} \in \mathcal{C}(\mu_1, \mu_2)} \int_{W_o(M) \times W_o(M)} d_\infty(\gamma, \eta) \hat{\pi}(d\gamma, d\eta) \leq W_{2,d_\infty}(\mu_1, \mu_2).$$

It is easy well-known that if $\delta > 1$ then the diffusion process is explosive so that (1.6) does not hold for any given $\lambda > 0$.

The remainder of the paper is organized as follows. In Section 2 we prove Theorem 1.1 under an additional assumption on bounded geometry (see (H) below), which in particular implies the regularity of finite-dimensional metrics induced by conditional expectations of the damped gradient. For readers' convenience to follow the main points of the proof, we address the proof of this regularity property in the Appendix at the end of the paper. Then a complete proof of Theorem 1.1 is presented in Section 3 by constructing Riemannian manifolds $\{(M_n, g_n) : n \geq 1\}$ and operators $\{L_n : n \geq 1\}$, which satisfy the assumption (H) and approximate the original Riemannian manifold and L in a good way. Since the intrinsic distance induced by the damped gradient on $W_o(M)$ is heavily dependent of the geometry of M , it is not consistent through our approximation. Finally, in Section 4 we extend Theorem 1.1 to the free path space.

2 The case with bounded geometry

In this section shall assume that

$$(H) \quad \text{Ric} \in C_b^1, \quad \Omega \in C_b^0 \text{ and } Z \in C_b^2.$$

It is known that under (H) the measure μ is equivalent to the Wiener measure (see [4]). It is

also known that the filtration generated by $\{u_s(w); s \leq t\}$ coincides with the one generated by $\{\gamma_s(w); s \leq t\}$; they are both equal to the natural filtration \mathcal{N}_t generated by $\{w_s; s \leq t\}$ (see [5, 17]). For $F \in \mathcal{FC}_b^\infty$, i.e.

$$(2.1) \quad F(\gamma) = f(\gamma_{s_1}, \dots, \gamma_{s_N}), \quad 0 < s_1 < \dots < s_N \leq T,$$

for some $N \geq 1$ and $f \in C_b^\infty(M^N)$, we define

$$(D_s F)(\gamma) = \sum_{j=1}^N u_{s_j}^{-1}(\partial_j f) \mathbf{1}_{\{s < s_j\}},$$

where ∂_j is the gradient with respect to the j -th component.

Throughout the paper, for any p -tensor \mathcal{T} on M , let $\mathcal{T}^\# : O(M) \rightarrow \mathcal{L}(\mathbb{R}^{d \times p}; \mathbb{R})$ with

$$\mathcal{T}^\#(u)(a_1, \dots, a_p) = \mathcal{T}(ua_1, \dots, ua_p), \quad a_1, \dots, a_p \in \mathbb{R}^d, u \in O(M).$$

Now, let $\text{Ric}_Z = \text{Ric} - \nabla Z$ and $\text{Ric}_Z^\#$ be defined for $\mathcal{T} = \text{Ric}_Z$. Consider the following resolvent equation on $\mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$:

$$(2.2) \quad \frac{dQ_{t,s}}{dt} = -\frac{1}{2} \text{Ric}_Z^\#(u_t) Q_{t,s}, \quad t \geq s > 0; \quad Q_{s,s} = \text{Id}.$$

By (1.1),

$$(2.3) \quad \|Q_{t-s}\| \leq e^{K(t-s)/2}, \quad t \geq s > 0,$$

where $\|\cdot\|$ is the operator norm on \mathbb{R}^d . Following [7], we define the damped gradient

$$(2.4) \quad (\tilde{D}_s F)(\gamma) = \sum_{j=1}^N Q_{s_j,s}^* u_{s_j}^{-1}(\partial_j f) \mathbf{1}_{\{s < s_j\}}, \quad F(\gamma) = f(\gamma_{s_1}, \dots, \gamma_{s_N}),$$

where $Q_{s_j,s}^*$ is the adjoint of $Q_{s_j,s}$. Then there holds the following integration by parts formula

$$(2.5) \quad \mathbb{E} \left(\int_0^T \langle \tilde{D}_s F, \dot{h}(s) \rangle ds \right) = \mathbb{E} \left(F \int_0^T \langle \dot{h}(s), dw_s \rangle \right), \quad h \in \mathbb{H}.$$

Indeed, letting \tilde{h} solve

$$(2.6) \quad \dot{\tilde{h}}(t) + \frac{1}{2} \text{Ric}_Z^\#(u_t) \tilde{h}(t) = \dot{h}(t), \quad \tilde{h}(0) = 0,$$

we have

$$\begin{aligned}
\int_0^T \langle D_s F, \dot{h}(s) \rangle ds &= \sum_{j=1}^N \langle u_{s_j}^{-1}(\partial_j f), \tilde{h}(s_j) \rangle = \sum_{i=1}^N \int_0^{s_j} \left\langle u_{s_j}^{-1}(\partial_j f), \frac{d}{ds} Q_{s_j, s} \tilde{h}(s) \right\rangle ds \\
&= \sum_{j=1}^N \int_0^{s_j} \langle u_{s_j}^{-1}(\partial_j f), Q_{s_j, s} \dot{h}(s) \rangle ds \\
&= \sum_{j=1}^N \int_0^T \langle Q_{s_j, s}^* u_{s_j}^{-1}(\partial_j f) \mathbf{1}_{\{s < s_j\}}, \dot{h}(s) \rangle ds = \int_0^T \langle \tilde{D}_s F, \dot{h}(s) \rangle ds.
\end{aligned}$$

Then (2.5) follows from the known integration by parts formula for the Malliavian gradient (see [2, 5, 7]). Under the hypothesis (H), we can use the approach [4] to get following log-Sobolev inequality (see also [6] for a possible degenerate diffusion)

$$(2.7) \quad \mu(F^2 \log F^2) \leq 2\mu\left(\int_0^T |\tilde{D}_s F|^2 ds\right), \quad F \in \mathcal{FC}_b^\infty, \mu(F^2) = 1.$$

Indeed, under our notations the last formula on page 75 of [4] (see Section 3 therein for the case with drift) becomes

$$H_t = \mathbb{E}\left(D_t F - \frac{1}{2} \int_t^T Q_{s,t}^* \text{Ric}_Z^\#(u_s) D_s F ds \middle| \mathcal{N}_t\right) = \mathbb{E}(\tilde{D}_t F | \mathcal{N}_t),$$

where the last equation follows from the above relationship between the gradient and the damped gradient. Then, replacing F by F^2 in the second formula on page 75 in [4] and noting that $\mathbb{E}F^2 = 1$, we obtain

$$\mathbb{E}F^2 \log F^2 \leq 2\mathbb{E} \int_0^T \frac{[\mathbb{E}(F \tilde{D}_t F | \mathcal{N}_t)]^2}{\mathbb{E}(F^2 | \mathcal{N}_t)} dt \leq 2\mathbb{E} \int_0^T |\tilde{D}_t F|^2 dt,$$

which is nothing but (2.7).

We shall derive the desired transportation-cost inequality from this log-Sobolev inequality. It was observed by [13] (see also [3, 20]) that the log-Sobolev inequality on a finite-dimensional manifold implies the corresponding transportation-cost inequality with respect to the intrinsic distance of the associated Dirichlet form, which has been recently extended in [15] to an abstract setting under certain assumption on the corresponding Hamilton-Jacobi semigroup. Since this assumption does not directly apply to our present situation, we shall adopt an approximation argument as in [20]. To this end, we first reduce (2.7) to a finite-dimensional log-Sobolev inequality, which implies a finite-dimensional transportation-cost inequality; then pass to the infinite-dimensional setting by taking limit with respect to a sequence of partitions of $[0, T]$. Note that the role of (2.7) is only intermediate here, used throughout the bounded geometry approximation; the constants involved will be well behaved when the uniform distance will be taken into account.

2.1 The finite-dimensional setting

Let $I = \{0 < s_1 \cdots < s_N \leq T\}$ be a partition of $[0, T]$. Let

$$\Lambda_I(\gamma) := (\gamma(s_1), \dots, \gamma(s_N)), \quad \gamma \in W_o(M)$$

be the projection from $W_o(M)$ onto the product manifold M^I . Then $\mu_I := (\Lambda_I)_* \mu$ has a smooth and strictly positive density with respect to the Riemannian volume $dx_1 \cdots dx_N$ on M^I . For $F = f \circ \Lambda_I \in \mathcal{FC}_b^\infty$, it follows from (2.4) that

$$(2.8) \quad \int_0^T |\tilde{D}_s F|^2 ds = \sum_{i,j=1}^N \int_0^{s_i \wedge s_j} \langle u_{s_j} Q_{s_j,s} Q_{s_i,s}^* u_{s_i}^{-1} \partial_i f, \partial_j f \rangle_g ds.$$

Let $M^I \ni z \mapsto \mathbb{P}(z, \cdot)$ be the regular conditional distributions of \mathbb{P} given $\Lambda_I \circ \gamma$. We define the linear operator $A^I(z)$ on $T_z M^I$ by

$$(2.9) \quad \langle A^I(z)X, Y \rangle_{g^I} = \int_{W_0(\mathbb{R}^d)} \left(\sum_{i,j=1}^N \int_0^{s_i \wedge s_j} \langle u_{s_j} Q_{s_j,s} Q_{s_i,s}^* u_{s_i}^{-1} X_i(z), Y_j(z) \rangle_g ds \right) d\mathbb{P}(z, \cdot)$$

for $X, Y \in T_z M^I$, where g^I is the product Riemannian metric on M^I and X_i and Y_j are the i -th and j -th components of X and Y respectively. By Propositions 5.4 and 5.5 below, A^I is uniformly positive definite and has a continuous version, denoted again by A^I . Moreover, since a continuous mapping on TM^I can be uniformly approximated by smooth ones, in the sequel we may and do assume that A^I is smooth.

Noting that for $F = f \circ \Lambda_I \in \mathcal{FC}_b^\infty$ we have

$$\mathbb{E} \left(\int_0^T |\tilde{D}_s F|^2 ds \right) = \int_{M^I} \langle A^I \nabla^I f, \nabla^I f \rangle_{g^I} d\mu_I,$$

where ∇^I is the gradient operator induced by g^I on M^I , it follows from (2.7) that

$$(2.10) \quad \mu_I(f^2 \log f^2) \leq 2\mu_I(\langle A^I \nabla^I f, \nabla^I f \rangle_{g^I}), \quad f \in C_b^\infty(M^I), \mu_I(f^2) = 1.$$

Now, let ρ_I be the Riemannian distance induced by A^I on M^I . We have

$$(2.11) \quad \rho_I(z, z') = \sup \{ |f(z) - f(z')| : f \in C_b^1(M^I), \langle A^I \nabla^I f, \nabla^I f \rangle_{g^I} \leq 1 \}.$$

Since g is complete, (H) and Proposition 5.5 below imply the completeness of ρ_I . Therefore, by [19, Theorem 1.1] with $p = 2$ (see also [13, 3]), (2.10) implies

$$(2.12) \quad W_{2,\rho_I}^2(f\mu_I, \mu_I) \leq 2\mu_I(f \log f), \quad f \geq 0, \mu_I(f) = 1.$$

We are now ready to prove the main result of the paper under the assumption (H).

Proposition 2.1. *Assume (1.1) and (H). Let $d_I(z, z') := \max_{1 \leq i \leq N} \rho(z_{s_i}, z'_{s_i})$, $z, z' \in M^I$. We have*

$$(2.13) \quad W_{2,d_I}^2(f\mu_I, \mu_I) \leq 2 \frac{e^{KT} - 1}{K} \mu_I(f \log f), \quad f \geq 0, \mu_I(f) = 1.$$

Proof. By (2.12), we it suffices to prove that

$$(2.14) \quad d_I^2 \leq \frac{e^{KT} - 1}{K} \rho_I^2.$$

Obviously,

$$(2.15) \quad d_I(z, z') = \sup \left\{ |f(z) - f(z')| : f \in C_b^\infty(M^I), \sum_{j=1}^N |\partial_j f|_g \leq 1 \right\}, \quad z, z' \in M^I.$$

Next, by (2.8) and the definition of A^I , we have

$$(2.16) \quad \rho_I(z, z') \geq \sup \left\{ |f(z) - f(z')| : f \in C_b^\infty(M^I), \int_0^T |\tilde{D}_s F|^2 ds \leq 1 \right\}$$

for $z, z' \in M^I$. Finally, for $f \in C_b^\infty(M^I)$ and $F = f \circ \Lambda_I$, (2.8) and (2.3) imply

$$\int_0^T |\tilde{D}_s F|^2 ds \leq \frac{e^{KT} - 1}{K} \sum_{i,j=1}^N |\partial_i f|_g |\partial_j f|_g = \frac{e^{KT} - 1}{K} \left(\sum_{j=1}^N |\partial_j f|_g \right)^2.$$

Therefore, (2.14) follows from (2.15) and (2.16). \square

2.2 The infinite-dimensional case

Proposition 2.2. *Assume (H). Then (1.1) implies (1.4).*

Proof. Since $\mathcal{F}C_b^\infty$ is dense in $L^1(\mu)$, it suffices to prove (1.4) for nonnegative $F \in \mathcal{F}C_b^\infty$ with $\mu(F) = 1$. Let $F = f \circ \Lambda_I$ for some partition I of $[0, T]$ and nonnegative $f \in C_b^\infty(M^I)$ with $\mu_I(f) = 1$. Take a sequence of partitions $\{I_n\}$ of $[0, T]$, which is finer and finer such that $I_n \supset I$ for all $n \geq 1$ and $\cup_{n \geq 1} I_n$ is dense in $[0, T]$. Let

$$\tilde{d}_{I_n}(\gamma, \eta) = d_{I_n}(\Lambda_{I_n}(\gamma), \Lambda_{I_n}(\eta)), \quad \gamma, \eta \in W_o(M).$$

Then

$$(2.17) \quad \tilde{d}_{I_n} \uparrow d_\infty \quad \text{as } n \uparrow \infty.$$

Since $I_n \supset I$, we may regard f as a function on M^{I_n} depending only on components in M^I so that $\mu_{I_n}(f) = 1$ and $\mu_{I_n}(f \log f) = \mu(F \log F)$ for all $n \geq 1$. By (2.13), for any $n \geq 1$, there exists a coupling measure $\tilde{\pi}_n \in C(f\mu_{I_n}, \mu_{I_n})$ such that (cf. [14])

$$(2.18) \quad \int_{M^{I_n} \times M^{I_n}} d_{I_n}^2 d\tilde{\pi}_n \leq 2 \frac{e^{KT} - 1}{K} \mu(F \log F).$$

For any $n \geq 1$, let $\mu(z, \cdot)$ (respectively $(F\mu)(z, \cdot)$) be the regular conditional distribution of μ (respectively $F\mu$) given $\Lambda_{I_n} = z \in M^{I_n}$. Then, according to [20, page 187] (see also [9, page 353]),

$$\hat{\pi}_n(d\gamma, d\eta) := \int_{M^{I_n} \times M^{I_n}} (F\mu)(z; d\gamma) \mu(z', d\eta) \tilde{\pi}_n(dz, dz') \in \mathcal{C}(F\mu, \mu), \quad n \geq 1.$$

Moreover, it is easy to see that (2.18) implies

$$(2.19) \quad \int_{W_o(M) \times W_o(M)} \tilde{d}_{I_n}^2 d\hat{\pi}_n \leq 2 \frac{e^{KT} - 1}{K} \mu(F \log F).$$

Since as explained on page 187 of [20] that $\mathcal{C}(F\mu, \mu)$ is tight and closed under the weak topology, up to a subsequence $\hat{\pi}_n \rightarrow \hat{\pi}$ weakly for some $\hat{\pi} \in \mathcal{C}(F\mu, \mu)$ as $n \rightarrow \infty$. Then, for any $N > 0$, it follows from (2.19) and the monotonicity of \tilde{d}_{I_n} in n that

$$\begin{aligned} \int_{W_o(M) \times W_o(M)} (\tilde{d}_{I_N}^2 \wedge N) d\hat{\pi} &= \lim_{n \rightarrow \infty} \int_{W_o(M) \times W_o(M)} (\tilde{d}_{I_N}^2 \wedge N) d\hat{\pi}_n \\ &\leq \int_{W_o(M) \times W_o(M)} \tilde{d}_{I_n}^2 d\hat{\pi}_n \leq 2 \frac{e^{KT} - 1}{K} \mu(F \log F). \end{aligned}$$

Therefore, the proof is finished by taking $N \rightarrow \infty$ and using (2.17). \square

3 Proof of Theorem 1.1

To prove Theorem 1.1 from Proposition 2.2, we shall constructed a sequence of metrics $\{g_n\}$ and drifts $\{Z_n\}$ satisfying (H) and $\text{Ric}_n - \nabla_n Z_n \geq -K_n$ with $K_n \rightarrow K$ and $\mu_n \rightarrow \mu$, where Ric_n, ∇_n are the Ricci curvature and the Levi-Civita connection induced by g_n , and μ_n is the distribution of the diffusion process generated by $L_n := \frac{1}{2}\Delta_n + Z_n$. Here, we will take g_n as conformal changes of g . So, we first study the conformal change of metric.

3.1 Conformal changes of metric for (H)

In this subset, we prove that the conformal change of metric used in [17] satisfies the assumption (H). More precisely, let $f \in C_0^\infty(M)$ with $0 \leq f \leq 1$ such that $M' := \{f > 0\}$ is a non-empty open set. Then, according to [17], M' is a complete Riemannian manifold under the metric $g' := f^{-2}g$, and

$$L' := f^2 L = \frac{1}{2} \Delta' + Z'$$

for $Z' = f^2 Z + \frac{d-2}{2} \nabla f^2$, where Δ' is the Laplacian induced by g' . Let $\mathcal{X}(M')$ be the set of all smooth vector fields on M' , and $\mathcal{X}_b^p(M', g')$ (respectively, $\mathcal{X}_b^p(M', g)$) the set of all C_b^p vector fields on M' with respect to the metric g' (respectively, g). Moreover, Let ∇' be the Levi-Civita connection on (M', g') . We have (see [1, Theorem 1.159 a)])

$$(3.1) \quad \nabla'_X Y = \nabla_X Y - \langle X, \log f \rangle_g X - \langle Y, \log f \rangle_g Y + \langle X, Y \rangle_g \nabla \log f, \quad X, Y \in \mathcal{X}(M').$$

Moreover, letting Ric' be the Ricci curvature on (M', g') , by [1, Theorem 1.159 d)] we have (note that the Laplacian therein equals to our $-\Delta$)

$$(3.2) \quad \begin{aligned} \text{Ric}' &= \text{Ric} - (d-2)(\text{Hess}_{\log f^{-1}} - (d \log f^{-1}) \otimes (d \log f^{-1})) \\ &\quad - (\Delta \log f^{-1} + (d-2)|\log f|_g^2)g \\ &= \text{Ric} + (d-2)f^{-1}\text{Hess}_f + (f^{-1}\Delta f - (d-3)|\nabla \log f|_g)g. \end{aligned}$$

Due to (3.1) and (3.2), we are able to prove the following main result in this subsection.

Proposition 3.1. *For $Z \in C^2$, the Riemannian manifold (M', g') and the drift $Z' := f^2 Z + \frac{d-2}{2} \nabla f^2$ satisfies (H).*

This Proposition will be implied by Lemma 3.3 and Lemma 3.4 below. To prove these lemmas, we first clarify the relationship between $\mathcal{X}_b^1(M', g)$ and $\mathcal{X}_b^1(M', g')$.

Lemma 3.2. *For any $X \in \mathcal{X}(M')$,*

$$(3.3) \quad \left| |\nabla X|_g - |\nabla' X|_{g'} \right| \leq 3|\nabla f|_g |X|_{g'}.$$

Consequently,

$$(3.4) \quad f\mathcal{X}_b^1(M', g) := \{fX : X \in \mathcal{X}_b^1(M', g)\} \subset \mathcal{X}_b^1(M', g') \subset \mathcal{X}_b^1(M', g).$$

Proof. For any $Y \in TM'$ with $|Y|_g = 1$, one has $|fY|_{g'} = 1$ and by (3.1),

$$\begin{aligned} \left| |\nabla_Y X|_g - |\nabla'_{fY} X|_{g'} \right| &= \left| |\nabla_Y X|_g - |\nabla'_Y X|_g \right| \leq |\nabla_Y X - \nabla'_Y X|_g \\ &\leq 3|f^{-1}X|_g |\nabla f|_g |Y| = 3|X|_{g'} |\nabla f|_g. \end{aligned}$$

Thus, (3.3) holds. Since f is smooth with $0 \leq f \leq 1$, it is obvious that

$$\mathcal{X}_b^0(M', g') = f\mathcal{X}_b^0(M, g) \subset \mathcal{X}_b^0(M, g),$$

where \mathcal{X}_b^0 denotes the set of all bounded continuous vector fields.

If $X \in \mathcal{X}_b^1(M', g)$, then (3.3) implies

$$|\nabla'(fX)|_{g'} \leq |\nabla(fX)|_g + 3|\nabla f|_g|X|_g \leq 4|\nabla f|_g|X|_g + f|\nabla X|_g,$$

which is bounded. So, $f\mathcal{X}_b^1(M, g) \subset \mathcal{X}_b^1(M', g')$.

On the other hand, if $X \in \mathcal{X}_b^1(M', g')$, then by (3.3),

$$|\nabla X|_g \leq |\nabla'X|_{g'} + 3|\nabla f|_g|X|_{g'}$$

is bounded. Therefore, the proof is finished. \square

Lemma 3.3. *For any C^2 vector field Z on M , one has $Z' \in \mathcal{X}_b^2(M', g')$.*

Proof. We shall prove $f^2Z \in \mathcal{X}_b^2(M', g')$ and $\nabla f^2 \in \mathcal{X}_b^2(M', g')$ respectively.

(a) For any $X \in \mathcal{X}_b^1(M', g')$, by (3.1) we have

$$\begin{aligned} \nabla'_X(f^2Z) &= \nabla_X(f^2Z) - \langle X, \nabla f \rangle_g fZ - \langle Z, \nabla f \rangle_g fX + \langle fZ, X \rangle_g \nabla f \\ &= f\{f\nabla_X Z + \langle Z, X \rangle_g \nabla f + \langle X, \nabla f \rangle_g Z - \langle Z, \nabla f \rangle_g X\} =: fU. \end{aligned}$$

By (3.4) we have $X \in \mathcal{X}_b^1(M', g)$. Moreover, $Z \in \mathcal{X}_b^2(M', g)$. Thus, $U \in \mathcal{X}_b^1(M', g)$. So, by (3.4), $\nabla'_X(f^2Z) \in \mathcal{X}_b^1(M', g')$ for all $X \in \mathcal{X}_b^1(M', g')$. This means that $f^2Z \in \mathcal{X}_b^2(M', g')$.

(b) Let $X \in \mathcal{X}_b^1(M', g')$, it remains to prove that $|\nabla'\nabla'_X \nabla f^2|_{g'}$ is bounded. By (3.1)

$$\begin{aligned} \nabla'_X \nabla f^2 &= \nabla_X \nabla f^2 - 2\langle X, \nabla f \rangle_g \nabla f - 2|\nabla f|_g^2 X + 2\langle \nabla f, X \rangle_g \nabla f \\ &= 2\langle X, \nabla f \rangle_g \nabla f + 2f\nabla_X \nabla f - 2|\nabla f|^2 X. \end{aligned}$$

By (3.4), $f\nabla_X \nabla f \in \mathcal{X}_b^1(M', g')$. So, it suffices to prove that

$$I := |\nabla'(\langle X, \nabla f \rangle_g \nabla f - |\nabla f|_g^2 X)|_{g'}$$

is bounded. By (3.3),

$$\begin{aligned} I &\leq |\nabla(\langle X, \nabla f \rangle_g \nabla f)|_g + |\nabla(|\nabla f|_g^2 X)|_g + 3|\nabla f|_g^2 |\langle f^{-1}X, \nabla f \rangle_g| + 3|\nabla f|_g^3 |X|_{g'} \\ &\leq 5|\nabla^2 f|_g |\nabla f|_g |X|_g + 5|\nabla f|_g^3 |X|_{g'} + 2|\nabla f|_g^2 |\nabla X|_g \end{aligned}$$

which is bounded since $X \in \mathcal{X}_b^1(M', g') \subset \mathcal{X}_b^1(M', g)$. \square

Lemma 3.4. *The Ricci curvature $\text{Ric}' \in C_b^1(M', g')$ and the curvature tensor $\Omega' \in C_b^0(M', g')$.*

Proof. By (3.2), there exists a smooth 2-tensor \mathcal{T} on M such that

$$\text{Ric}'(X, Y) = f \mathcal{T}(f^{-1}X, f^{-1}Y), \quad X, Y \in \mathcal{X}_b^1(M', g').$$

Then R' is bounded since $|\cdot|_g = f|\cdot|_{g'}$. Assuming $|X|_{g'} = |Y|_{g'} = 1$, we obtain from the above formula and (3.3) that

$$\begin{aligned} |\nabla' \text{Ric}'(X, Y)|_{g'} &= f |\nabla \text{Ric}'(X, Y)|_g \\ &\leq f |\nabla f|_g |\mathcal{T}|_g + f |\nabla \mathcal{T}|_g (|\nabla(f^{-1}X)|_g + |\nabla(f^{-1}Y)|_g) \\ &\leq f |\nabla f|_g |\mathcal{T}|_g + |\nabla \mathcal{T}|_g (|\nabla X|_g + |\nabla Y|_g + 2|\nabla f|_g), \end{aligned}$$

which is bounded on M' since $X, Y \in \mathcal{X}_b^1(M', g') \subset \mathcal{X}_b^1(M', g)$, \mathcal{T} is smooth on M , and $M' \subset M$ is relatively compact. Therefore $\text{Ric}' \in C_b^1(M', g')$. The same argument does work for Ω' . The proof is finished. \square

3.2 Proof of Theorem 1.1

By Greene-Wu's approximation theorem [11], we take a smooth positive function $\tilde{\rho}$ on M such that

$$(3.5) \quad |\tilde{\rho} - \rho_o| \leq 1, \quad \frac{1}{2} \leq |\nabla \tilde{\rho}| \leq 2 \quad \text{and} \quad (\Delta + Z)\tilde{\rho} \leq (\Delta + Z)\rho_o + 1,$$

where the last inequality is restricted outside $\{o\} \cup \text{cut}(o)$. Moreover, by the approximation theorem, we may and do assume that $Z \in C^2$.

Lemma 3.5. *(1.1) implies*

$$(\Delta + Z)\tilde{\rho} \leq K + 2 + \psi(\tilde{\rho} + 1), \quad \rho \geq 1.$$

Proof. For $x \notin \text{cut}(o)$ with $\rho_o = \rho_o(x) \geq 1$, let $\ell : [0, \rho_o] \rightarrow M$ be the minimal geodesic from o to x . Let $U = \dot{\ell}$ and $\{U_i : 1 \leq i \leq d-1\}$ be constant vector fields along ℓ such that $\{U, U_i : 1 \leq i \leq d-1\}$ is an orthonormal basis. Let J_i be the Jacobi field along ℓ with $J_i(0) = 0$ and $J_i(\rho_o) = U_i, 1 \leq i \leq d$. Let $h(s) = 1 - (\rho_o - s)^+$. By the second variational formula and the index lemma, we have

$$\begin{aligned} (3.6) \quad \Delta \rho_o(x) &= \sum_{i=1}^{d-1} \int_0^{\rho_o} \{|\nabla_U J_i|_g^2 - R(J_i, U, J_i, U)\} \\ &\leq \sum_{i=1}^{d-1} \int_0^{\rho_o} \{|\nabla_U (hU_i)|_g^2 - h^2 R(U_i, U, U_i, U)\} = 1 - \int_0^{\rho_o} h^2 \text{Ric}(U, U). \end{aligned}$$

Next,

$$Z\rho_o = \langle Z, U \rangle_g(x) = \int_0^{\rho_o} \frac{d}{ds} \{h^2 \langle Z, U \rangle_g\} ds \leq \int_0^{\rho_o} h^2 \langle \nabla_U Z, U \rangle_g + \psi \circ \rho_o(x).$$

Combining this with (3.6) we obtain

$$(\Delta + Z)\rho_o \leq K + 1 + \psi \circ \rho_o.$$

Therefore, the proof is finished by (3.5). \square

Proof of Theorem 1.1. Let $h_0 \in C_b^\infty$ be decreasing such that $0 \leq h_0 \leq 1$, $h_0(s) = 1$ for $s \leq 1$, and $h_0(s) = 0$ for $s \geq 2$. Let

$$h_n(s) := h_0\left(\frac{1}{n} \int_0^s \frac{dt}{\psi(t+1)}\right), \quad s \geq 0, n \geq 2.$$

For any $n \geq 2$, let $f_n = h_n(\tilde{\rho})$. Since $\psi > 0$ is smooth with $\int_0^\infty \psi(s)^{-1} ds = \infty$, we have $f_n \in C_0^\infty(M)$. Let μ_n be the distribution on $W_o(M)$ for the diffusion process generated by $f_n^2 L$. Then $\mu_n \rightarrow \mu$ strongly; that is, for any bounded measurable function F on $W_o(M)$,

$$(3.7) \quad \lim_{n \rightarrow \infty} \mu_n(F) = \mu(F).$$

Indeed, letting τ_n be the hitting time of the L -diffusion process to the set $\{\int_0^{\tilde{\rho}} \psi(s)^{-1} ds \geq n\}$, these two diffusion processes have the same distribution up to τ_n . So,

$$|\mu(F) - \mu_n(F)| \leq 2\|F\|_\infty \mathbb{P}(\tau_n \leq T).$$

Since $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$, we obtain (3.7). Then, it is standard that

$$(3.8) \quad W_{2,d_\infty}^2(F\mu, \mu) \leq \liminf_{n \rightarrow \infty} W_{2,d_\infty}^2(F_n\mu_n, \mu_n)$$

for $F_n := F/\mu_n(F)$.

Now, let Ric^n, ∇^n be the Ricci curvature and Levi-Civita connection induced by $g_n := f_n^{-2}g$ on $M_n := \{f_n > 0\}$. Let $Z_n = f_n^2 Z + (d-2)f_n \nabla f_n$. By (3.8), Propositions 2.2 and 3.1, it remains to prove

$$(3.9) \quad \text{Ric}^n - \nabla^n Z_n \geq -K_n$$

for some positive constants $K_n \rightarrow K$ as $n \rightarrow \infty$. Let $X \in TM_n$ with $|X|_{g_n} = 1$. By (3.2), we have

$$\text{Ric}^n(f_n U, f_n U) = f_n^2 \text{Ric}(U, U) + (d-2)f_n \text{Hess}_f(U, U) + f_n \Delta f_n - (d-3)|\nabla f_n|^2, \quad |U|_g = 1.$$

Combining this with the first display on [17, page 114], we obtain

$$\begin{aligned} & (\text{Ric}^n - \nabla^n Z_n)(f_n U, f_n U) \\ & \geq f_n^2 (\text{Ric} - \nabla Z)(U, U) + f_n (\Delta + Z)f_n - c_1 (|\nabla f_n|_g^2 + |Z|_g |\nabla f_n|_g), \quad |U|_g = 1 \end{aligned}$$

for some constant $c_1 > 0$. Combining this with (1.1) we obtain

$$\text{Ric}^n - \nabla^n Z_n \geq -K + f_n (\Delta + Z)f_n - c_1 (|\nabla f_n|_g^2 + |Z|_g |\nabla f_n|_g).$$

Therefore, to ensure (3.9) it suffices to show that

$$(3.10) \quad \lim_{n \rightarrow \infty} \inf \{f_n (\Delta + Z)f_n - c_1 (|\nabla f_n|_g^2 + |Z|_g |\nabla f_n|_g)\} = 0.$$

By Lemma 3.5, $h'_0 \leq 0$ and $|\nabla \tilde{\rho}| \leq 2$,

$$(\Delta + Z)f_n \geq -\frac{\|h'_0\|_\infty (K + 2 + \psi(\tilde{\rho} + 1))}{n\psi(\tilde{\rho} + 1)} - \frac{2\|h''_0\|_\infty}{n^2\psi(\tilde{\rho} + 1)^2}$$

which goes to zero uniformly as $n \rightarrow \infty$. Similarly, $|\nabla f_n|_g^2 + |Z|_g |\nabla f_n|_g \rightarrow 0$ uniformly too. \square

4 An extension to free path spaces

Let ν be a probability measure on M such that

$$(4.1) \quad W_{2,\rho}(f\nu, \nu)^2 \leq C_0 \nu(f \log f), \quad f \geq 0, \nu(f) = 1$$

holds for some constant $C_0 > 0$. Let P_ν be the distribution of the L -diffusion process starting from ν up to time $T > 0$, which is then a probability measure on the free path space $W(M) = C([0, T]; M)$.

Theorem 4.1. *Under (1.1) and the growth condition for $|Z|$ stated in Theorem 1.1 for some (and hence any) fixed point $o \in M$. Then*

$$W_{2,d_\infty}(FP_\nu, P_\nu)^2 \leq \left(C_0 e^{KT} + 2 \frac{e^{KT} - 1}{K}\right) P_\nu(F \log F), \quad F \geq 0, P_\nu(F) = 1.$$

Proof. (a) Without loss of generality, we assume that $F \in \mathcal{FC}_b^\infty$ is strictly positive. Let P_x be the distribution of the L -diffusion process starting from x , and let $f(x) = P_x(F)$, $F_x = \frac{F}{f(x)}$. Then $\nu(f) = P_x(F_x) = 1$ and

$$(4.2) \quad P_{f\nu} = \int_M (F_x P_x) f(x) \nu(dx), \quad P_{f\nu} = \int_M P_x f(x) \nu(dx).$$

. By the triangle inequality,

$$(4.3) \quad W_{2,d_\infty}(FP_\nu, P_\nu) \leq W_{2,d_\infty}(FP_\nu, P_{f\nu}) + W_{2,d_\infty}(P_{f\nu}, P_\nu).$$

(b) It is well-known that in a class of probability measures on a Polish space with bounded second moment, the weak convergence is equivalent to the convergence in the L^2 Wasserstein distance (see e.g. [14]). Noting that $x \mapsto P_x$ and $x \mapsto F_x P_x$ are continuous in the weak topology for probability measures on $W(M)$, and due to (1.1), $\sup_x P_x(e^{d_\infty(x,\cdot)}) < \infty$, we conclude that

$$x \mapsto W_{2,d_\infty}(P_x, F_x P_x)$$

is continuous. Furthermore, Theorem 1.1 and the uniform boundedness of F_x imply that this function is bounded. Therefore, it is to see from (4.2) that

$$(4.4) \quad W_{2,d_\infty}(FP_\nu, P_{f\nu})^2 \leq \int_M W_{2,d_\infty}(F_x P_x, P_x)^2 f(x) \nu(dx).$$

Indeed, letting $\{A_{i,n} : i \geq 1\}_{n \geq 1}$ be a sequence of measurable partitions of M such that

$$\nu(A_{i,n}) + \text{dia}(A_{i,n}) \leq \frac{1}{n}, \quad i, n \geq 1,$$

where $\text{dia}(A_{i,n})$ is the diameter of $A_{i,n}$. By the continuity of f , let $x_{i,n} \in \bar{A}_{i,n}$ such that

$$f(x_{i,n})\nu(A_{i,n}) = \int_{A_{i,n}} f(x) \nu(dx), \quad i, n \geq 1.$$

Let $\pi_{i,n} \in \mathcal{C}(F_{x_{i,n}} P_{x_{i,n}}, P_{x_{i,n}})$ such that

$$\int_{W(M) \times W(M)} d_\infty^2 d\pi_{i,n} = W_{2,d_\infty}(F_{x_{i,n}} P_{x_{i,n}}, P_{x_{i,n}})^2, \quad i, n \geq 1.$$

Then

$$\pi_n := \sum_{i=1}^{\infty} f(x_{i,n})\nu(A_{i,n})\pi_{i,n} \in \mathcal{C}((FP_\nu)_n, (P_{f\nu})_n),$$

where

$$(FP_\nu)_n := \sum_{i=1}^{\infty} f(x_{i,n})\nu(A_{i,n})F_{x_{i,n}}P_{x_{i,n}} \rightarrow FP_\nu$$

and

$$(P_{f\nu})_n := \sum_{i=1}^{\infty} f(x_{i,n})\nu(A_{i,n})P_{x_{i,n}} \rightarrow P_{f\nu}$$

weakly as $n \rightarrow \infty$, then

$$\begin{aligned} W_{2,d_\infty}(FP_\nu, P_{f\nu})^2 &= \lim_{n \rightarrow \infty} W_{2,d_\infty}((FP_\nu)_n, (P_{f\nu})_n)^2 \\ &\leq \lim_{n \rightarrow \infty} \int_{W(M) \times W(M)} d_\infty^2 d\pi_n \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} f(x_{i,n})\nu(A_{i,n})W_{2,d_\infty}(F_{x_{i,n}}P_{x_{i,n}}, P_{x_{i,n}})^2 \\ &= \int_M W_{2,d_\infty}(F_x P_x, P_x)^2 f(x)\nu(dx), \end{aligned}$$

Therefore, (4.4) holds. Combining this with Theorem 1.1, we obtain

$$\begin{aligned} (4.5) \quad W_{2,d_\infty}(FP_\nu, P_{f\nu})^2 &\leq \frac{2(e^{KT} - 1)}{K} \int_{W(M) \times M} \{F_x(\gamma) \log F_x(\gamma)\} P_x(d\gamma) f(x)\nu(dx) \\ &= \frac{2(e^{KT} - 1)}{K} (P_\nu(F \log F) - \nu(f \log f)). \end{aligned}$$

(c) To estimate $W_{2,d_\infty}(P_{f\nu}, P_\nu)$, let $\hat{\pi} \in (f\nu, \nu)$ such that

$$W_{2,\rho}(f\nu, \nu)^2 = \int_{M \times M} \rho^2 d\hat{\pi},$$

and let (X_t, Y_t) be the coupling by parallel displacement for the L -diffusion process with initial distribution $\hat{\pi}$. By [21, (3.2)] (note that the present L is half of the one therein)

$$\rho(X_t, Y_t) \leq \rho(X_0, Y_0)e^{Kt/2}, \quad t > 0.$$

Thus,

$$W_{2,d_\infty}(P_{f\nu}, P_\nu)^2 \leq \mathbb{E} \max_{t \in [0, T]} \rho(X_t, Y_t)^2 \leq e^{KT} \mathbb{E} \rho(X_0, Y_0)^2 = e^{KT} W_{2,\rho}(f\nu, \nu)^2.$$

Then it follows from (4.1) that

$$W_{2,d_\infty}(P_{f\nu}, P_\nu)^2 \leq C_0 e^{KT} \nu(f \log f).$$

Combining this with (4.3) and (4.5) we arrive at

$$\begin{aligned} W_{2,d_\infty}(FP_\nu, P_\nu)^2 &\leq (1+\delta)W_{2,d_\infty}(FP_\nu, P_{f\nu})^2 + (1+\delta^{-1})W_{2,d_\infty}(P_{f\nu}, P_\nu)^2 \\ &\leq \frac{2(1+\delta)(e^{KT}-1)}{K} P_\nu(F \log F) + \left(C_0(1+\delta^{-1})e^{KT} - \frac{2(1+\delta)(e^{KT}-1)}{K} \right) \nu(f \log f). \end{aligned}$$

Then the proof is finished by taking $\delta = C_0 K e^{KT} / 2(e^{KT} - 1)$. \square

5 Appendix: regularity of A^I

Let V be a smooth manifold. For the convenience of our exposition, we shall introduce V -valued smooth Wiener functional in the following way (for a general definition, we refer to [12], p.78). Let $\Phi : W_o(\mathbb{R}^d) \rightarrow V$ be a measurable map. Let $p > 1$. We say that Φ is derivable if there exists $\nabla\Phi(w) \in \mathbb{H} \otimes T_{\Phi(w)}V$ satisfying $\mathbb{E}(\|\nabla\Phi\|_{\mathbb{H} \otimes TV}^p) < +\infty$ such that for each $h \in \mathbb{H}$, Φ admits a version Φ_h such that $\varepsilon \mapsto \Phi_h(w + \varepsilon h)$ is C^1 and

$$\frac{d}{d\varepsilon} \Phi_h(w + \varepsilon h)|_{\varepsilon=0} = \nabla\Phi(w) \cdot h \in T_{\phi(w)}V.$$

Then $\nabla\Phi$ is a map from $W_o(\mathbb{R}^d)$ into $\mathbb{H} \otimes TV$. Inductively, we define high order derivatives $\nabla^k\Phi : W_o(\mathbb{R}^d) \rightarrow \mathbb{H}^{\otimes k} \otimes TV$. We say that $\Phi \in \mathbb{D}_k^\infty$ if $\mathbb{E}(\|\nabla^r\Phi\|^p) < +\infty$ for all $r \leq k$ and $p > 1$. We say that Φ is non-degenerated in Malliavin sense if $\det^{-1}[\nabla\Phi(\nabla\Phi)^*] \in \cap_{p>1} L^p$, where $(\nabla\Phi(w))^* : T_{\Phi(w)}V \rightarrow \mathbb{H}$ is defined by

$$\langle (\nabla\Phi(w))^*v, h \rangle_H = \langle \nabla\Phi(w)h, v \rangle_{T_{\Phi(w)}V}.$$

The following result holds (see [12], chapter III).

Theorem 5.1. *Let $\Phi \in \mathbb{D}_2^\infty$ be a V -valued non-degenerated Wiener functional and $G \in \mathbb{D}_1^\infty(W_o(\mathbb{R}^d), \mathbb{R})$, then the conditional expectation $z \mapsto \mathbb{E}(G|\Phi = z)$ admits a continuous version.*

Now we are going to prove the regularity of A^I .

Lemma 5.2. *Assume (H). Then the Itô functional $u_t : W_o(\mathbb{R}^d) \rightarrow O(M)$ defined by (1.2) belongs to \mathbb{D}_1^∞ .*

Proof. We first note that for any $h \in \mathbb{H}$, the law of $w \mapsto u_t(w + \varepsilon h)$ is equivalent to that of u_t and furthermore,

$$\beta(t) := \langle \theta, D_h u_t \rangle, \quad \rho(t) := \langle \Theta, D_h u_t \rangle$$

satisfy (see [2, (2.21)])

$$(5.1) \quad \begin{cases} d\beta(t) = (\dot{h}(t) + \{\nabla Z\}^\#(u_t) \beta(t) - \rho(t) Z^\#(u_t)) dt + \rho(t) (Z^\#(u_t) dt + \circ dw_t), \\ d\rho(t) = \Omega_{u_t}(u_t^{-1} Z_{\pi u_t} dt + \circ dw_t, \beta(t)). \end{cases}$$

Here, $Z^\#(u) := \langle Z, u \cdot \rangle \in \mathbb{R}^d$ for $u \in O(M)$, (θ, Θ) is the parallelism of $O(M)$, an $\mathbb{R}^d \times \mathfrak{so}(d)$ -valued one-form on $O(M)$ defined by

$$\theta_u(\tilde{X}) = u^{-1} \pi^* \tilde{X}, \quad \Theta_u(\tilde{X}) = q_u^{-1}(P_V \tilde{X}), \quad u \in O(M), \tilde{X} \in T_u O(M),$$

where P_V is the orthogonal projection from $TO(M)$ onto the space of vertical tangent vectors on $O(M)$, and

$$q_u : \mathfrak{so}(d) \ni \alpha \mapsto \frac{d}{ds} \{ u e^{-s\alpha} \} |_{s=0} \in P_V T_u O(M)$$

is an endomorphism.

Let $D(d) = \mathbb{R}^d \times \mathfrak{so}(d)$. For $r \in O(M)$, we denote by $\mathcal{M}_j(u)$ the endomorphism of $D(d)$ defined by

$$(x, A) \mapsto (\{\nabla Z\}^\#(u) \cdot x + A e_j, \Omega_r(e_j, x)).$$

Let $J_{t,s}$ solve the equation on $\mathcal{L}(\mathbb{R}^d; \mathbb{R}^d \times \mathfrak{so}(d))$:

$$(5.2) \quad \frac{d}{dt} J_{t,s} = \left(\sum_{j=1}^d \mathcal{M}_j(u_t) \circ [(u_t^{-1} Z_{\pi u_t})^j dt + dw_t^j] \right) J_{t,s}, \quad t > s, \quad J_{s,s} = (\text{Id}_{D(d)}, 0).$$

Then (see [12, page 292])

$$(\beta(t), \rho(t)) = \int_0^t J_{t,s} \dot{h}(s) ds.$$

This completes the proof due to the fact that

$$\left| \frac{d}{d\varepsilon} u_t(w + \varepsilon h) \right|_{\varepsilon=0} \Big|_{T_{u_t} O(M)}^2 = |\beta(t)|^2 + |\rho(t)|^2$$

and the boundedness of \mathcal{M}, Ω and Z . □

Lemma 5.3. *Assume (H). Let $Q_t = Q_{t,0}$. Then*

$$(5.3) \quad D_h Q_t = -\frac{1}{2} \int_0^t Q_{t,s} \{ \nabla_{\pi^* D_h u_s} \text{Ric}_Z \}^\#(u_s) Q_s ds, \quad h \in \mathbb{H}.$$

Consequently, $Q_t \in \mathbb{D}_1^p(W_0(\mathbb{R}^d))$ for $p \geq 1$.

Proof. Differentiating (2.2), we obtain

$$\frac{dD_h Q_t}{dt} = -\frac{1}{2}\{\nabla_{\pi^* D_h u_t} \text{Ric}_Z\}^\#(u_t)Q_t - \frac{1}{2}(\text{Ric}_Z^\#(u_t))D_h Q_t, \quad D_h Q_0 = 0.$$

So, we get the expression (5.3) and thus, $Q_t \in \mathbb{D}_1^p(W_0(\mathbb{R}^d))$ for $p \geq 1$ due to (H) and Lemma 5.2. \square

Proposition 5.4. *Assume (H). Then $A^I : TM^I \rightarrow TM^I$ has a continuous μ_I -version.*

Proof. Let $K_{ij}(s) = Q_{s_j, s} Q_{s_i, s}^*$. Note that $u_{s_i}^{-1} X_i(\gamma(s_i)) = \theta(X_i^\#)_{u_{s_i}}$. Then, for any compactly supported smooth vector fields X, Y on M^I ,

$$\langle u_{s_j} Q_{s_j, s} Q_{s_i, s}^* u_{s_i}^{-1} X_i, Y_j \rangle_g = \langle K_{ij}(s) \theta(X_i^\#)_{u_{s_i}}, \theta(X_j^\#)_{u_{s_j}} \rangle := G_{ij}(t).$$

By lemmas 5.2 and 5.3, $G_{ij}(t)$ are in $\mathbb{D}_1^\infty(W_0(\mathbb{R}^d), \mathbb{R})$, so

$$G := \sum_{i,j=1}^N \int_0^{s_i \wedge s_j} G_{ij}(s) ds \in \mathbb{D}_1^\infty(W_0(\mathbb{R}^d), \mathbb{R}).$$

By Theorem 5.1, $z \mapsto \langle A^I(z)X(z), Y(z) \rangle_{g^I} = \int_{W_0(\mathbb{R}^d)} G \mathbb{P}(z, dw)$ has a continuous version. \square

Proposition 5.5. *Assume that $\text{Ric} - \nabla Z \leq K_1$, then A^I is uniformly elliptic with respect to g^I .*

Proof. Let $a = (a_1, \dots, a_N) \in T_z M^I$. Suppose without losing the generality, that $|a_N| = \max_{1 \leq i \leq N} |a_i|$. Take (X_1, \dots, X_N) be vector fields around (z_1, \dots, z_N) such that

$$(X_1(z_1), \dots, X_N(z_N)) = (a_1, \dots, a_N).$$

We have

$$\langle A^I(z)a, a \rangle = \mathbb{E}_\mu \left(\int_0^T \left| \sum_{j=1}^N Q_{s_j, s}^* (u_{s_j}^{-1} X_j(\gamma(s_j))) \mathbf{1}_{(s < s_j)} \right|^2 ds \middle| \Lambda_I = z \right).$$

Let $s_{N-1} \leq s < s_N$ and $v \in \mathbb{R}^d$. Then by the assumption on the upper bound of Ric,

$$\frac{d}{dt} |Q_{t,s} v|^2 \geq -K_1 |Q_{t,s} v|^2.$$

It follows that $|Q_{t,s} v|^2 \geq e^{-K_1(t-s)} |v|^2 \geq e^{-K_1(s_N - s_{N-1})} |v|^2$. Therefore,

$$\begin{aligned}
& \int_0^T \left| \sum_{j=1}^N Q_{s_j, s}^*(u_{s_j}^{-1} X_j(\gamma(s_j)) \mathbf{1}_{(s < s_j)}) \right|^2 ds \\
& \geq \int_{s_{N-1}}^{s_N} |Q_{s_N, s}^*(u_{s_N}^{-1} X_N(\gamma(s_N)))|^2 ds \\
& \geq |X_N(\gamma(s_N))|^2 e^{-K_1(s_N - s_{N-1})} (s_N - s_{N-1}).
\end{aligned}$$

Hence

$$\langle A^I(z)a, a \rangle \geq |a_N|^2 e^{-K_1(s_N - s_{N-1})} (s_N - s_{N-1}) \geq |a|^2 N^{-1} e^{-K_1(s_N - s_{N-1})} (s_N - s_{N-1}).$$

□

Acknowledgements The authors would like to thank the referee for useful comments on an earlier version of the paper.

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